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On the nonlinear reflexion of a gravity wave at a critical level. Part 1

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In this paper we examine the nonlinear interaction of a forced internal gravity wave in a stratified fluid with its critical level. The representative Richardson number J is taken to be large and the undisturbed state consists of a hyperbolic-tangent velocity profile and an almost constant density gradient. It is assumed that at large values of a non-dimensional time t the flow outside the critical layer is steady, consisting of the mean shear together with a disturbance periodic in x that corresponds to the single harmonic of the incident wave of small amplitude ϵ. The requirements of a match across the critical layer lead to a reflected wave and a transmitted wave both of whose amplitudes are $O(\epsilon e^{-\nu\pi})$ when $1 \ll t \ll \epsilon^{-\frac{2}{3}}$, where $\nu = (J-\frac{1}{4})^{\frac{1}{2}}$. For $\nu \gg 1$ the layer therefore acts as a wave absorber, and the purpose of this investigation is to ascertain whether this property persists on an even longer time scale. At times $t = O(\epsilon^{-\frac{2}{3}})$ the layer has thickness $O(\epsilon^{\frac{2}{3}})$ and the first few terms of an expansion in powers of $\epsilon^{\frac{2}{3}}t$ show that higher harmonics are forced on the outer flow, and the reflexion and transmission coefficients develop with time. The leading-order correction to these coefficients is calculated explicitly; that to the transmission coefficient is again exponentially small in ν though that to the reflexion coefficient is $O(\nu^{-1})$. The reflexion coefficient is therefore increasing and the critical layer begins to restore wave energy to the outer flow. Owing to the complexity of the calculation higher-order corrections are not obtained here, but the results presented are in agreement with predictions of earlier workers that the layer acts as an absorber and a reflector but not as a transmitter.

1. Introduction

It is well known that an internal gravity wave, propagating through a stratified shear flow, develops singular characteristics at the critical level where its phase speed is equal to the mean horizontal velocity of the undisturbed fluid. The phenomenon is associated by dynamical meteorologists with the notion of energy trapping by the troposphere and by oceanographers with the confining of topographically generated disturbances to the lower region of the ocean and of wind-generated disturbances to the upper region. Important contributions to the theoretical understanding of the structure of the flow near the critical layer were made in papers by Bretherton (1966) and Booker & Bretherton (1967) in which linear aspects were considered. Bretherton showed that when the Richardson number is arbitrarily large a wave packet moving with the local group velocity does not reach its critical level in a finite time and so is neither transmitted nor reflected but is absorbed. Further details are given by Grimshaw (1975) and Hartman (1975). Booker & Bretherton (1967) solved an initial-value problem for a single sinusoidal component forced at the lower boundary and found that the amplitude of the upward-propagating wave is reduced by a factor proportional to $\exp\{-\pi(J-\frac{1}{4})^{\frac{1}{2}}\}$, J being a representative Richardson number. Presumably the momentum flux of the wave is absorbed at the critical layer and a corresponding force exerted on the mean flow.

Several authors (including Booker and Bretherton) have pointed out that in the absence of wave dissipation the nonlinear terms must become important in the critical layer after a sufficiently long time while still explicitly negligible elsewhere in the flow field. Our aim in the current work is to study the effect of this nonlinearity of the critical layer on the absorption and reflexion of an internal gravity wave maintained at an infinite distance above its critical layer. Our starting point is a linear quasi-steady state in which the motion in most of the flow field is steady, the time dependence being confined to the critical layer. The shear profile is a hyperbolic tangent and the density gradient is essentially constant. The Boussinesq approximation is applied. It is assumed that the quasi-steady state will establish itself when the effect of some prescribed initial condition has died out. The Richardson number J is greater than unity and this linear solution is in accord with the theory of Booker & Bretherton (1967) with both reflexion and transmission coefficients having a factor $\exp\{-\pi(J-\frac{1}{4})^{\frac{1}{2}}\}$, and this is very small even for moderate sized J. However this state cannot persist because the velocity and temperature in the critical layer are increasing without limit. At longer times the previously neglected nonlinear terms take effect and we wish to examine the reaction of the critical layer to them and in particular their effect on the reflexion and transmission coefficients. The method of attack is the same as was outlined in the Rossby wave problem discussed by Stewartson (1978) and used by the present authors (Brown & Stewartson 1978) to develop an analysis for free oscillations of a marginally stable flow with $J = \frac{1}{4}$. The former paper consisted of an analytical solution of a special case of the problem treated numerically by Warn & Warn (1976, 1978) and by Béland (1976), all of whom, following Dickinson (1970), made use of the property that the thickness of the critical layer is proportional to t^{-1}. The paper by Brown & Stewartson (1978) will subsequently be referred to as I. In I an expansion in the critical layer was formed in powers of $\tau\,(=\epsilon^{\frac{1}{2}}t)$, where ϵ was the amplitude of the free oscillation and t the time. Higher harmonics are forced on the outer flow and the calculation was taken to the point where the second harmonic first appeared, and a subsequent mismatch between the outer flow and the critical layer led to a Stuart–Landau type equation for the amplitude of the fundamental harmonic. In the present problem a similar expansion is formulated with ϵ now the amplitude of the forced incident wave, though because of its complexity the calculation is not taken so far. The appropriate time is again $\tau = \epsilon^{\frac{1}{2}}t$ and, as identified by Maslowe (1972), the thickness of the critical layer is $O(\epsilon^{\frac{1}{2}})$. The chief result is the correction to the first harmonic when $J \gg 1$ which leads to a correction $O(\tau^{3+2i\nu})$, where $\nu = (J-\frac{1}{4})^{\frac{1}{2}}$, to the reflexion coefficient and an $O(\tau^3)$ correction to the transmission coefficient. As a function of ν the transmission coefficient is again $O(e^{-\nu\pi})$. However the correction to the reflexion coefficient is $O(\nu^{-1})$, indicating a probable increase of amplitude of the reflected wave as time goes on. That the transmission coefficient will exhibit similar behaviour at later stages of the expansion will be demonstrated in a subsequent study. To the order considered here, there is no transfer of momentum flux to the mean shear, a phenomenon predicted by many earlier workers, but no doubt it will occur at a later stage of the expansion procedure. The present work is capable of providing further terms in powers of ν^{-1} and indeed may be extended to general values of J. The Boussinesq approximation is made but this is for simplicity rather than because it is fundamental. The density profile is chosen so that there is an exact analytic solution to the linear problem for all J and while this is not necessary it does help to give confidence that the asymptotic form is indeed correct for $\nu \gg 1$. The results are expected to be representative of a large class of similar problems when a wave is forced in a stratified shear flow not only at infinity but also on a finite boundary. If, however, terms of higher order in τ are required the present approach would seem to involve an unreasonable amount of analysis while, if terms of higher order in ϵ are required, nonlinear effects in the remainder of the flow field, i.e. outside the critical layer, must be taken into account.

There have been previous, mainly numerical, nonlinear studies of internal gravity waves (Breeding 1971; Klemp & Lilly 1978; Fritts 1978, 1979) and of their interaction with a critical level. Considerable difficulties were experienced by these authors in obtaining adequate resolution of the critical layer. This is one of the justifications for the present analytical approach, limited though its results are. Our findings are broadly in agreement with those of these numerical workers, to the extent that they all find some evidence of reflexion and little of transmission.

2. The basic equations

The situation is similar to, but not identical with, that considered in I. We again consider an inviscid shear layer separating two parallel streams of fluid in motion, the velocity in each stream being uniform but different. However since these streams are to be capable of sustaining plane waves the density gradients must be non-zero in each stream. We choose orthogonal Cartesian axes Ox^*y^* with origin in the centre of the shear layer, Ox^* parallel to the direction of the two streams, and O moving along the x^* axis with their mean velocity. The non-uniform densities of the streams mean that rather more care is required in justifying the use of the Oberbeck–Boussinesq approximation. We again take the equation of state to be linear of the form

$$\rho^* = \rho_0^*\{1-\beta^* T_0^*(T^*/T_0^*-1)\}, \tag{2.1}$$

where an asterisk denotes a physical variable, ρ^*, T^* are the density and temperature, β^* is the coefficient of volume expansion, and ρ_0^*, T_0^* denote a constant reference density and temperature. Then Mihaljan (1962) has shown that if

$$\beta^* T_0^* \ll 1 \tag{2.2}$$

the governing equations may be taken as

$$\operatorname{div} \mathbf{q}^* = 0, \tag{2.3}$$

$$\frac{\partial \mathbf{q}^*}{\partial t^*} + (\mathbf{q}^* . \nabla)\,\mathbf{q}^* = -\frac{\nabla p^*}{\rho_0^*} + \beta^* g^* (T^* - T_0^*)\,\nabla y^*, \tag{2.4}$$

$$\frac{\partial T^*}{\partial t^*} + \mathbf{q}^* . \nabla T^* = 0. \tag{2.5}$$

Here $\mathbf{q}^*$ is the velocity, t^* the time, p^* the pressure and g^* the acceleration due to gravity.

We now define the velocity difference between the two streams to be $2V^*$, choose a reference length L^* and write

$$x = x^*/L^*, \quad y = y^*/L^*, \quad t = t^*V^*/L^*, \tag{2.6}$$

$$\mathbf{q}^*/V^* = U(y)\nabla x + \epsilon\mathbf{q}(x,y,t), \tag{2.7}$$

$$\rho^*/\rho_0^* = 1 + \beta^* T_0^* R(y) + \epsilon\rho(x,y,t). \tag{2.8}$$

In (2.7), (2.8) the small positive number ϵ will layer be identified with the amplitude of the imposed incoming plane wave, and $U(y)$, $R(y)$ are the undisturbed non-dimensional velocity and density of the shear layer. Then if the non-dimensional temperature perturbation is $\epsilon T(x,y,t)$ the equation of state (2.1) becomes

$$\rho = -\beta^* T_0^* T, \tag{2.9}$$

and (2.3) to (2.5) lead to

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi - U''(y)\frac{\partial\psi}{\partial x} - \epsilon\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = -J\frac{\partial T}{\partial x}, \tag{2.10}$$

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)T + R'(y)\frac{\partial\psi}{\partial x} - \epsilon\frac{\partial(\psi,T)}{\partial(x,y)} = 0. \tag{2.11}$$

Here $\psi(x,y,t)$ is the perturbation stream function with

$$\mathbf{q} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right) \tag{2.12}$$

and

$$J = \beta^* T_0^* g^* L^*/V^{*2} > 1 \tag{2.13}$$

is the Richardson number.

An analytic solution of the linear equations is possible only for certain choices of the properties of the shear layer and in order to achieve this and to simplify the non-linear study we take

$$U(y) = \tanh y, \quad R'(y) = -1 + \frac{2}{J}\operatorname{sech}^2 y \tanh^2 y. \tag{2.14}$$

Then both streams have the same constant density gradient and the Oberbeck–Boussinesq approximation is formally justified by condition (2.2). A further comment on the choice of the density profile in (2.14) will be made in §4. When J is large, the situation of most interest in later sections, the density gradient is essentially constant throughout the flow.

3. Propagation directions of plane waves far from the critical layer

When $|y|$ is large the velocity and density profiles lead to $U(y) = \operatorname{sgn} y$ and $R'(y) = -1$, which are both constant. The linearized forms of (2.10), (2.11) reduce to

$$\left(\frac{\partial}{\partial t} + \operatorname{sgn} y\frac{\partial}{\partial x}\right)^2\nabla^2\psi + J\frac{\partial^2\psi}{\partial x^2} = 0, \tag{3.1}$$

which has plane wave solutions of the form

$$\psi \propto e^{i(\alpha x+my-\omega t)}, \tag{3.2}$$

where α, m, ω are constants satisfying

$$(\alpha \operatorname{sgn} y-\omega)^2 = J\alpha^2/(\alpha^2+m^2). \tag{3.3}$$

Outside the critical layer we envisage a quasi-steady disturbance in which for $y > 0$ there is a forced wave of the type (3.2) but with $\omega = 0$. The energy of the wave is partly absorbed by the critical layer, and is partly reflected and transmitted as waves of the same type except that the amplitudes may now depend algebraically on t. The appropriate conditions on α, m for such waves can be ascertained by studying an initial-value problem and making use of the concept of group velocity. The point has been discussed at length by Booker & Bretherton (1967) and we may summarize the results as follows:

When $\alpha m > 0$ a solution of (3.1) with an exponential factor

$$e^{i(\alpha x-my)}, \quad \alpha^2+m^2 = J \tag{3.4}$$

for $y \gg 1$ represents a wave travelling in the direction of x increasing and y decreasing, and so may be regarded as a wave incident on the critical layer. We shall regard this wave as given.

Again with $\alpha m > 0$ a solution of (3.1) with an exponential factor

$$e^{i(\alpha x+my)} \tag{3.5}$$

for $y \gg 1$ represents a wave travelling in the direction of x increasing and y increasing, and so may be regarded as a wave reflected upwards from the critical layer. Finally a wave with an exponential factor

$$e^{i(\alpha x+my)}, \quad \alpha m > 0 \tag{3.6}$$

for $y \ll -1$ represents a wave travelling in the direction of x decreasing and y decreasing and so may be regarded as a wave transmitted below the critical layer. The purpose of the present paper is to investigate the properties of the coefficients of (3.5) and (3.6) when the nonlinear effects of the critical layer are taken into account. The fourth solution, like (3.4) except that $y \ll -1$, represents a wave incident on the critical layer from below and is not relevant to our studies.

The solution of the problem when the imposed incident wave is below, rather than above, the critical layer, can be obtained from that discussed here on replacing x, y, T by $-x$, $-y$, $-T$ respectively. This is because firstly (2.10), (2.11) are unaltered by the transformation when (2.14) is taken into account, and secondly (3.4) becomes a wave incident below the critical layer and (3.5), (3.6) are again reflected and transmitted waves respectively.

4. Solution properties at finite values of y

The main effort of our investigation is concentrated in the immediate neighbourhood of $y = 0$ where the phase velocity of the imposed disturbance is equal to the velocity of the basic shear flow. Since the imposed wave is steady this velocity is zero. We shall be considering large values of J and the dominant properties of the disturbance when

$y \neq 0$ can be computed for general values of $U(y)$ by the method of steepest descents. However the reflexion and transmission coefficients involve powers of $e^{-\nu\pi}$, $\nu = (J-\frac{1}{4})^{\frac{1}{2}}$ that are strictly negligible in this asymptotic method, and so in order to ensure that no errors occur due to the omission of such terms it is convenient to have at our disposal a complete solution of the linearized equations in the outer part of the interaction region where $y = O(1)$. It is for this reason that the special but representative forms for $U(y)$, $R'(y)$ in (2.14) were selected. Then equations (2.10), (2.11), when linearized by setting $\epsilon = 0$, have a solution in which, with c.c. denoting the complex conjugate,

$$\psi = e^{i\alpha x}\phi_1(y) + \text{c.c.}, \quad (\phi_1'' - \alpha^2\phi_1)\tanh^2 y + J\phi_1 = 0. \tag{4.2}$$

The boundaiy conditions on ϕ_1 are fixed by the requirement that there is an incoming disturbance of prescribed amplitude when y is large and positive and only an outgoing disturbance when y is large and negative. This implies that, on use of the results of the preceding section,

$$\begin{aligned} \phi_1(y) &\approx \mathscr{R}_{11}e^{imy} + e^{-imy} \quad &\text{as} \quad y \to \infty, \\ \phi_1(y) &\approx \mathscr{T}_{11}e^{-im|y|} \quad &\text{as} \quad y \to -\infty, \end{aligned} \tag{4.3}$$

where $m\,(>0) = (J-\alpha^2)^{\frac{1}{2}}$ and $\mathscr{R}_{11}, \mathscr{T}_{11}$ are the reflexion and transmission coefficients. We are specially interested here in the properties of these when the nonlinear evolution of the critical layer is taken into account.

The general solution of (4.2) is

$$\phi_1(y) = A_{11}\phi_{11}(y) + A_{12}\phi_{12}(y), \tag{4.4}$$

where

$$\phi_{11}(y) = \frac{2^{im-1}}{\pi i}\int_C q^{2z}\frac{(-z-\frac{3}{4}-\frac{1}{2}i\nu)!\,(-z-\frac{3}{4}+\frac{1}{2}i\nu)!\,(z-1-\frac{1}{2}im)!\,(-im)!}{(-\frac{3}{4}-\frac{1}{2}i\nu-\frac{1}{2}im)!\,(-\frac{3}{4}+\frac{1}{2}i\nu-\frac{1}{2}im)!\,(-z-\frac{1}{2}im)!}\,dz, \tag{4.5}$$

$$\phi_{12}(y) = \frac{2^{-im-1}}{\pi i}\int_C q^{2z}\frac{(-z-\frac{3}{4}-\frac{1}{2}i\nu)!\,(-z-\frac{3}{4}+\frac{1}{2}i\nu)!\,(z-1+\frac{1}{2}im)!\,(im)!}{(-\frac{3}{4}-\frac{1}{2}i\nu+\frac{1}{2}im)!\,(-\frac{3}{4}+\frac{1}{2}i\nu+\frac{1}{2}im)!\,(-z+\frac{1}{2}im)!}\,dz, \tag{4.6}$$

$q = \sinh|y|$, C is a contour parallel to the imaginary axis of z and passing through a point on the real axis in the interval $(0, \frac{1}{4})$. Here and henceforth $i = e^{\frac{1}{2}i\pi}$ and ν, m are both positive. The function $z!$ is defined for complex values of z by analytic continuation from its definition for real values. When $|y|$ is large we complete the contour to the left and obtain

$$\phi_{11}(y) = e^{im|y|}(1+O(e^{-2|y|})), \quad \phi_{12}(y) = e^{-im|y|}(1+O(e^{-2|y|})). \tag{4.7}$$

When $|y|$ is small we complete the contour to the right and obtain

$$\phi_{11}(y) \approx \alpha_{11}|y|^{\frac{1}{2}+i\nu} + \beta_{11}|y|^{\frac{1}{2}-i\nu}, \quad \phi_{12}(y) \approx \alpha_{12}|y|^{\frac{1}{2}+i\nu} + \beta_{12}|y|^{\frac{1}{2}-i\nu}, \tag{4.8}$$

where

$$\left.\begin{aligned} \alpha_{11} &= \frac{(-i\nu-1)!\,(-im)!\,2^{im}}{(-\frac{1}{4}-\frac{1}{2}i\nu-\frac{1}{2}im)!\,(-\frac{3}{4}-\frac{1}{2}i\nu-\frac{1}{2}im)!}, \\ \beta_{11} &= \frac{(i\nu-1)!\,(-im)!\,2^{im}}{(-\frac{1}{4}+\frac{1}{2}i\nu-\frac{1}{2}im)!\,(-\frac{3}{4}+\frac{1}{2}i\nu-\frac{1}{2}im)!}. \end{aligned}\right\} \tag{4.9}$$

For α_{12}, β_{12}, it is only necessary to change the sign of m in (4.9). The relative error in (4.8) is $O(y^2)$ as $y \to 0$.

In particular when J is large for fixed α

$$\nu = J^{\frac{1}{2}} + O(J^{-\frac{1}{2}}) = m, \tag{4.10}$$

and so
$$\alpha_{11} \approx 2^{i\nu}, \quad \beta_{11} \approx -i2^{\frac{1}{2}+i\nu}e^{-\nu\pi}, \quad \alpha_{12} \approx i2^{\frac{1}{2}-i\nu}e^{-\nu\pi}, \quad \beta_{12} \approx 2^{-i\nu}. \tag{4.11}$$

The solution must now be completed by matching across the critical layer with

$$\left.\begin{aligned} \phi_1(y) &= \mathscr{R}_{11}\phi_{11}(y) + \phi_{12}(y) \quad \text{if} \quad y > 0, \\ \phi_1(y) &= \mathscr{T}_{11}\phi_{12}(y) \qquad\qquad\quad \text{if} \quad y < 0, \end{aligned}\right\} \tag{4.12}$$

so that
$$\phi_1(y) \approx (\mathscr{R}_{11}\alpha_{11} + \alpha_{12})\,y^{\frac{1}{2}+i\nu} + (\mathscr{R}_{11}\beta_{11} + \beta_{12})\,y^{\frac{1}{2}-i\nu} \quad \text{as} \quad y \to 0^+, \tag{4.13}$$

$$\phi_1(y) \approx \mathscr{T}_{11}\alpha_{12}|y|^{\frac{1}{2}+i\nu} + \mathscr{T}_{11}\beta_{12}|y|^{\frac{1}{2}-i\nu} \quad \text{as} \quad y \to 0^-. \tag{4.14}$$

The reflexion and transmission coefficients are determined by the match with the solution that holds in the neighbourhood of $y = 0$. In the following section this solution is obtained and the matching completed.

5. Linearized theory of the neighbourhood of $y = 0$

It is clear from (4.8) that the assumption of a linear theory must eventually fail in the neighbourhood of $y = 0$ since the x component of velocity tends to infinity as $y \to 0$. Consideration of an initial-value problem leads us to expect that the linear solution will be valid at any finite time for sufficiently small values of ϵ but will break down as $t \to \infty$ for any given ϵ. At large values of t the solution in the neighbourhood of $y = 0$ satisfies

$$\left(\frac{\partial}{\partial t} + i\alpha y\right)^2 \frac{\partial^2 \Phi}{\partial y^2} - \alpha^2 J \Phi = 0, \tag{5.1}$$

from (2.10), (2.11) with $\psi = e^{i\alpha x}\Phi(y,t) + \text{c.c.}$ and $U(y)$, $R'(y)$ replaced by their leading-order terms in the critical layer where $\partial/\partial y \gg \partial/\partial x$. The boundary conditions for (5.1) are to be chosen so that Φ matches with ϕ_1 given by (4.13), (4.14) as $|y| \to \infty$ in some sense. The appropriate solution of (5.1) is

$$\Phi = \frac{B_1}{(-\frac{3}{2} - i\nu)!} {}^{*}\!\!\int_0^{\alpha t} \frac{e^{-iyu}}{u^{\frac{1}{2}+i\nu}}\,du + \frac{B_2}{(-\frac{3}{2} + i\nu)!} {}^{*}\!\!\int_0^{\alpha t} \frac{e^{-iyu}}{u^{\frac{1}{2}-i\nu}}\,du, \tag{5.2}$$

where the $*$ denotes that the finite part of the integrals is to be taken and B_1, B_2 are constants so that Φ matches with ϕ_1 in (4.13), (4.14). It follows at once that the thickness of the critical layer is $O(t^{-1})$ and so the matching is justified provided t is large. As $ty \to +\infty$

$$\Phi \approx e^{\frac{1}{4}i\pi - \frac{1}{2}\nu\pi} y^{\frac{1}{2}+i\nu} B_1 + e^{\frac{1}{4}i\pi + \frac{1}{2}\nu\pi} y^{\frac{1}{2}-i\nu} B_2 \tag{5.3}$$

while
$$\Phi \approx e^{-\frac{1}{4}i\pi + \frac{1}{2}\nu\pi}|y|^{\frac{1}{2}+i\nu} B_1 + e^{-\frac{1}{4}i\pi - \frac{1}{2}\nu\pi}|y|^{\frac{1}{2}-i\nu} B_2 \tag{5.4}$$

as $ty \to -\infty$. Hence by comparison with (4.13), (4.14) we have

$$B_1 = \frac{\alpha_{12}(\alpha_{11}\beta_{12} - \alpha_{12}\beta_{11})}{\alpha_{11}\beta_{12}e^{2\nu\pi} - \alpha_{12}\beta_{11}} e^{-\frac{1}{4}i\pi + \frac{1}{2}\nu\pi}, \quad B_2 = \frac{\beta_{12}}{\alpha_{12}} e^{\nu\pi} B_1,$$

$$\mathscr{T}_{11} = \frac{e^{-\frac{1}{4}i\pi + \frac{1}{2}\nu\pi}}{\alpha_{12}} B_1, \quad \mathscr{R}_{11} = \frac{\alpha_{12}\beta_{12}(e^{2\nu\pi} - 1)}{\alpha_{12}\beta_{11} - \alpha_{11}\beta_{12}e^{2\nu\pi}}. \tag{5.5}$$

In particular when J is large it follows from (4.11) that

$$B_1 \approx 2^{\frac{1}{2}-i\nu} e^{\frac{1}{4}i\pi-\frac{1}{2}\nu\pi}, \quad B_2 \approx 2^{-i\nu} e^{-\frac{1}{4}i\pi-\frac{1}{2}\nu\pi},$$
$$\mathscr{T}_{11} \approx -i e^{-\nu\pi}, \qquad \mathscr{R}_{11} \approx -i 2^{\frac{1}{2}-2i\nu} e^{-\nu\pi}. \tag{5.6}$$

These results on the transmission and reflexion coefficients are, of course, well known, having originally been obtained by Booker & Bretherton (1967) and by many authors since. The usual interpretation is that when J is large the critical layer absorbs the incident wave. In fact from a practical point of view J does not have to be very large to make the reflexion and transmission coefficients essentially zero. Not surprisingly therefore the velocity and temperature in the critical layer rise rapidly in magnitude. For example at $y = 0$ we have from (5.2) that

$$\frac{\partial \psi}{\partial y} \approx \frac{-iB_2(\alpha t)^{\frac{1}{2}+i\nu} e^{i\alpha x}}{(-\frac{3}{2}+i\nu)!\,(\frac{1}{2}+i\nu)} + \text{c.c.} \approx \frac{(\alpha t)^{\frac{1}{2}+i\nu}}{(2\pi)^{\frac{1}{2}}} (2\nu)^{-i\nu} e^{i(\alpha x+\nu-\frac{3}{4}\pi)} + \text{c.c.} \tag{5.7}$$

when J is large so that it increases without limit with t. The corresponding temperature on the centre-line is i/ν times the expression in (5.7). In order to assess the importance of nonlinear effects the behaviour of the stream function itself is of more significance; we have from (5.2) again that

$$\psi \approx (\alpha t)^{-1}\,\partial \psi / \partial y \tag{5.8}$$

and so $|\psi| \sim t^{-\frac{1}{2}}$ as $t \to \infty$. Hence the nonlinear terms in (2.10), (2.11) are significant in the critical layer when $\epsilon t^{\frac{3}{2}} \sim 1$, and at these times the outer solution, where $y = O(1)$, is still controlled by the linear terms. We shall now investigate how the reflexion and transmission coefficients are modified by these terms in the critical layer.

6. The nonlinear equations of the critical layer

Since as shown in the preceding section both the horizontal component of velocity and the temperature become large with t the linear theory eventually fails. The time scale on which it does so is $O(\epsilon^{-\frac{2}{3}})$ and the critical layer then has thickness $O(\epsilon^{\frac{2}{3}})$. If in (2.10), (2.11) we write

$$y = \epsilon^{\frac{2}{3}} Y, \quad \tau = \epsilon^{\frac{2}{3}} \alpha t, \quad \psi = \epsilon^{\frac{1}{3}} \Psi(x, Y, \tau), \quad T = \epsilon^{-\frac{1}{3}} S(x, Y, \tau), \tag{6.1}$$

the appropriate equations are, in the limit $\epsilon = 0$,

$$\left(\alpha \frac{\partial}{\partial \tau} + Y \frac{\partial}{\partial x}\right) \frac{\partial^2 \Psi}{\partial Y^2} + J \frac{\partial S}{\partial x} = \frac{\partial \Psi}{\partial x} \frac{\partial^3 \Psi}{\partial Y^3} - \frac{\partial \Psi}{\partial Y} \frac{\partial^3 \Psi}{\partial x \partial Y^2}, \tag{6.2}$$

$$\left(\alpha \frac{\partial}{\partial \tau} + Y \frac{\partial}{\partial x}\right) S - \frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial x} \frac{\partial S}{\partial Y} - \frac{\partial \Psi}{\partial Y} \frac{\partial S}{\partial x}. \tag{6.3}$$

The initial conditions for these equations are specified by the requirement that the solution shall match, as $\tau \to 0$, with the linearized solutions of § 5 which we envisage to be the form taken for large t of the solution of an initial-value problem starting at $t = 0$. Thus $\tau = O(1)$ represents the next stage in the development of the flow and we shall find that on this time scale the outer flow, where $y = O(1)$, although still linear is no longer steady.

When J is large we see from (5.6) that $B_1/B_2 = O(e^{-2\nu\pi})$ and so the first term in

(5.2) is negligible compared with the second, and in order to make the subsequent analysis tractable we shall retain the second only. Thus for our initial conditions for (6.2), (6.3) we take

$$\Psi = e^{i\alpha x}\Psi_{11}(Y,\tau)+\text{c.c.},\quad S = \frac{e^{i\alpha x}}{\frac{1}{2}-i\nu}\frac{\partial\Psi_{11}}{\partial Y}+\text{c.c.} \tag{6.4}$$

as $\tau \to 0$, where

$$\Psi_{11}(Y,\tau) = b\int_0^\tau \frac{e^{-iYu}}{u^{\frac{3}{2}-i\nu}}du, \tag{6.5}$$

and

$$b = B_2\epsilon^{-\frac{3}{4}i\nu}/(-\tfrac{3}{2}+i\nu)!. \tag{6.6}$$

This means that when ν is large

$$b \approx \frac{\epsilon^{-\frac{3}{4}i\nu}}{(2\pi)^{\frac{1}{2}}}\nu^{1-i\nu}2^{-i\nu}e^{i(\nu+\frac{1}{4}\pi)}. \tag{6.7}$$

In addition there will be matching conditions outside the critical layer which will be altered from its steady form given in § 4 as the critical layer forces the intrusion of higher harmonics and the development of the coefficients of the first harmonic as functions of τ. This matching, which is straightforward, is outlined in § 7.

The procedure now parallels closely that of I. We develop a formal expansion of Ψ, S in powers of τ, whose coefficients are functions of $Y\tau, x, \epsilon^{i\nu}, \tau^{i\nu}$ and shall find the first few terms explicitly.

We write

$$\Psi = \sum_{r=1}^{\infty}\Psi_r(Y,\tau,x),\quad S = \sum_{r=1}^{\infty}S_r(Y,\tau,x), \tag{6.8}$$

where Ψ_r, S_r are of the form

$$\Psi_r = \sum_{n=-r}^{r} e^{ni\alpha x}\Psi_{rn}(Y,\tau),\quad S_r = \sum_{n=-r}^{r} e^{ni\alpha x}S_{rn}(Y,\tau) \tag{6.9}$$

and

$$\Psi_{10} = S_{10} = 0,\quad S_{11} = (\tfrac{1}{2}-i\nu)^{-1}\partial\Psi_{11}/\partial Y. \tag{6.10}$$

Also

$$\tilde{\Psi}_{r,n} = \Psi_{r,-n},\quad \tilde{S}_{rn} = S_{r,-n}, \tag{6.11}$$

the complex conjugates being denoted by tildes, and where without loss of generality we may take $r-n$ to be an even integer or zero. As in I each Ψ_r, S_r is in magnitude $\tau^{\frac{3}{2}}$ times Ψ_{r-1}, S_{r-1} and is of the form $\tau^{-2+\frac{3}{2}r}$ multiplied by a function of $Y\tau, \tau^{i\nu}, \epsilon^{i\nu}$.

On substituting (6.8), (6.9) into (6.2), (6.3) we find that Ψ_{rn}, S_{rn} satisfy

$$\left(\frac{\partial}{\partial\tau}+inY\right)\frac{\partial^2\Psi_{rn}}{\partial Y^2}+inJS_{rn} = G_{rn}(Y,\tau), \tag{6.12}$$

$$\left(\frac{\partial}{\partial\tau}+inY\right)S_{rn}-in\Psi_{rn} = H_{rn}(Y,\tau), \tag{6.13}$$

where G_{rn}, H_{rn} are known functions depending on the previously calculated Ψ_p, S_p with $1 \leqslant p \leqslant r-1$. If S_{rn} is eliminated between (6.12), (6.13) then $\overline{\Psi}_{rn}$, the Laplace transform of Ψ_{rn}, satisfies, on denoting all transforms by an overbar,

$$(s+inY)^2\frac{\partial^2\overline{\Psi}_{rn}}{\partial Y^2}-n^2J\overline{\Psi}_{rn} = (s+inY)\,\overline{G}_{rn}(Y,s)-inJ\overline{H}_{rn}(Y,s). \tag{6.14}$$

The solution of (6.14) is

$$\bar{\Psi}_{rn} = \frac{1}{2n\nu}(s+inY)^{\frac{1}{2}-i\nu}\int_{-\infty}^{Y}\bar{K}_{rn}(Y_1,s)\,(s+inY_1)^{-\frac{1}{2}+i\nu}\,dY_1$$
$$-\frac{1}{2n\nu}(s+inY)^{\frac{1}{2}+i\nu}\int_{-\infty}^{Y}\bar{K}_{rn}(Y_1,s)\,(s+inY_1)^{-\frac{1}{2}-i\nu}\,dY_1$$
$$+\bar{C}_{rn}(s+inY)^{\frac{1}{2}-i\nu}+\bar{D}_{rn}(s+inY)^{\frac{1}{2}+i\nu}, \quad (6.15)$$

where $\bar{K}_{rn}(Y,s)$ has been written for the right-hand side of (6.14) and $\bar{C}_{rn}(s)$, $\bar{D}_{rn}(s)$ are to be found by matching with the outer solution. It follows from (6.15) that as $Y \to -\infty$

$$\Psi_{rn} \approx C_{rn}(n|Y|)^{\frac{1}{2}-i\nu}e^{-\frac{1}{4}i\pi-\frac{1}{2}\nu\pi}+D_{rn}(n|Y|)^{\frac{1}{2}+i\nu}e^{-\frac{1}{4}i\pi+\frac{1}{2}\nu\pi}, \quad (6.16)$$

while as $Y \to +\infty$

$$\Psi_{rn} \approx (C_{rn}+I_{rn})\,(nY)^{\frac{1}{2}-i\nu}e^{\frac{1}{4}i\pi+\frac{1}{2}\nu\pi}+(D_{rn}-J_{rn})\,(nY)^{\frac{1}{2}+i\nu}e^{\frac{1}{4}i\pi-\frac{1}{2}\nu\pi}, \quad (6.17)$$

where
$$I_{rn} = \frac{1}{2n\nu(\frac{1}{2}-i\nu)!}\int_0^{\tau}(\tau-\tau_r)^{\frac{1}{2}-i\nu}\,d\tau_r\int_{-\infty}^{\infty}K_{rn}(Y,\tau_r)\,e^{-inY(\tau-\tau_r)}\,dY, \quad (6.18)$$

$$J_{rn} = \frac{1}{2n\nu(\frac{1}{2}+i\nu)!}\int_0^{\tau}(\tau-\tau_r)^{\frac{1}{2}+i\nu}\,d\tau_r\int_{-\infty}^{\infty}K_{rn}(Y,\tau_r)\,e^{-inY(\tau-\tau_r)}\,dY. \quad (6.19)$$

The reflexion and transmission coefficients of the outer solution where $y = O(1)$ will not be affected until a non-zero I_{rn} or J_{rn} is obtained, at which stage the matching condition will be non-homogeneous. The method of performing the match is outlined in the following section.

7. Development of the outer solution

On the time scale $t = O(\epsilon^{-\frac{2}{3}})$ the outer solution is linear with ψ of order unity. Since we have retained only the leading-order terms in ϵ in the inner region it is sufficient to do likewise in the outer, in which case ψ will be of the form

$$\psi = e^{ni\alpha x}\psi_n(y,\tau)+\text{c.c.}, \quad \left(\frac{\partial^2\psi_n}{\partial y^2}-n^2\alpha^2\psi_n\right)\tanh^2 y+J\psi_n = 0. \quad (7.1)$$

The neglected derivative with respect to t is of relative order $\epsilon^{\frac{2}{3}}$ and the neglected correction to the mean flow forced by the critical layer turns out to be of relative order $\epsilon^{\frac{1}{3}}$, see (8.7), (8.8) below. The solution with $n = 1$ at $\tau = 0$ is that given in § 4, so that $\psi_1(y,0) = \phi_1(y)$ and $\phi_1(y)$ appears in (4.4). The values of the reflexion and transmission coefficients at $\tau = 0$ appear in (5.6). As τ increases these depend on τ as does $\psi_1(y,\tau)$, this behaviour being forced by the critical layer as is the appearance of the higher harmonics.

The general solution of (7.1) is

$$\psi_n(y,\tau) = A_{n1}(\tau)\,\phi_{n1}(y)+A_{n2}(\tau)\,\phi_{n2}(y), \quad (7.2)$$

where ϕ_{n1}, ϕ_{n2} may be obtained from (4.5), (4.6) on replacing m by m_n, where

$$m_n = (J-n^2\alpha^2)^{\frac{1}{2}}.$$

The connexion between the behaviours of ϕ_{n1}, ϕ_{n2} for large and small $|y|$ may be obtained in the same manner from (4.7) to (4.9).

The boundary conditions on ψ_n are again fixed by the requirement that there is an incoming disturbance of prescribed τ-independent amplitude when y is large and positive, and only an outgoing disturbance if y is large and negative. On use of the results of § 3 this implies that

$$\left.\begin{aligned}\psi_n(y,\tau) &\approx \mathscr{R}_n(\tau)\, e^{im_n y} + \delta_{n1} e^{-im_n y} \quad \text{as} \quad y \to \infty,\\ \psi_n(y,\tau) &\approx \mathscr{T}_n(\tau)\, e^{-im_n|y|} \quad \text{as} \quad y \to -\infty,\end{aligned}\right\} \tag{7.3}$$

where $m_1 = m$. By analogy with (4.12) we now have

$$\left.\begin{aligned}\psi_n(y,\tau) &= \mathscr{R}_n(\tau)\,\phi_{n1}(y) + \delta_{n1}\phi_{n2}(y) \quad \text{if} \quad y > 0,\\ \psi_n(y,\tau) &= \mathscr{T}_n(\tau)\,\phi_{n2}(y) \quad \text{if} \quad y < 0.\end{aligned}\right\} \tag{7.4}$$

If we now write

$$\mathscr{R}_n(\tau) = \sum_{r=n}^{\infty} \mathscr{R}_{rn}(\tau), \quad \mathscr{T}_n(\tau) = \sum_{r=n}^{\infty} \mathscr{T}_{rn}(\tau), \tag{7.5}$$

where $\mathscr{R}_{rn}, \mathscr{T}_{rn}$ are of the form $\tau^{\frac{3}{2}(r-1)}$ times a function of $\tau^{i\nu}, \epsilon^{i\nu}$ then $\mathscr{R}_{11}, \mathscr{T}_{11}$ are independent of τ and are given by (5.6).

We are now in a position to match with the solution of § 6. From (6.16) to (6.19) and (7.4), (7.5) we obtain

$$\mathscr{R}_{rn}\alpha_{n1} + \delta_{r1}\delta_{n1}\alpha_{n2} = \epsilon^{-\frac{3}{2}i\nu} n^{\frac{1}{2}+i\nu} e^{\frac{1}{4}i\pi-\frac{1}{2}\nu\pi}(D_{rn} - J_{rn}), \tag{7.6}$$

$$\mathscr{R}_{rn}\beta_{n1} + \delta_{r1}\delta_{n1}\beta_{n2} = \epsilon^{\frac{3}{2}i\nu} n^{\frac{1}{2}-i\nu} e^{\frac{1}{4}i\pi+\frac{1}{2}\nu\pi}(C_{rn} + I_{rn}), \tag{7.7}$$

$$\mathscr{T}_{rn}\alpha_{n2} = \epsilon^{-\frac{3}{2}i\nu} n^{\frac{1}{2}+i\nu} e^{-\frac{1}{4}i\pi+\frac{1}{2}\nu\pi} D_{rn}, \tag{7.8}$$

$$\mathscr{T}_{rn}\beta_{n2} = \epsilon^{\frac{3}{2}i\nu} n^{\frac{1}{2}-i\nu} e^{-\frac{1}{4}i\pi-\frac{1}{2}\nu\pi} C_{rn}, \tag{7.9}$$

for the four unknowns $\mathscr{R}_{rn}, \mathscr{T}_{rn}, C_{rn}, D_{rn}$. When $r = n = 1$, $J_{11} = I_{11} = 0$ and the solutions for $\mathscr{R}_{11}, \mathscr{T}_{11}$ are as in (5.5). Also

$$B_1 = \epsilon^{-\frac{3}{2}i\nu} D_{11}, \quad B_2 = \epsilon^{\frac{3}{2}i\nu} C_{11}. \tag{7.10}$$

When r and n are not both unity the solution of (7.6) to (7.9) is

$$\mathscr{R}_{rn} = -e^{\frac{1}{4}i\pi-\frac{1}{2}\nu\pi}\left\{\frac{\epsilon^{-\frac{3}{2}i\nu} n^{\frac{1}{2}+i\nu} J_{rn}\beta_{n2} + \epsilon^{\frac{3}{2}i\nu} n^{\frac{1}{2}-i\nu} e^{-\nu\pi} I_{rn}\alpha_{n2}}{\alpha_{n1}\beta_{n2} - \alpha_{n2}\beta_{n1} e^{-2\nu\pi}}\right\}, \tag{7.11}$$

$$\mathscr{T}_{rn} = -e^{-\frac{1}{4}i\pi-\frac{1}{2}\nu\pi}\left\{\frac{\epsilon^{-\frac{3}{2}i\nu} n^{\frac{1}{2}+i\nu} e^{-\nu\pi} J_{rn}\beta_{n1} + \epsilon^{\frac{3}{2}i\nu} n^{\frac{1}{2}-i\nu} I_{rn}\alpha_{n1}}{\alpha_{n1}\beta_{n2} - \alpha_{n2}\beta_{n1} e^{-2\nu\pi}}\right\}, \tag{7.12}$$

from which C_{rn}, D_{rn} follow on use of (7.8), (7.9).

In the next section we obtain the first non-zero I_{rn}, J_{rn}. We find that

$$I_{22} = J_{22} = I_{33} = J_{33} = 0$$

but that I_{31}, J_{31} are non-zero. It emerges that $J_{31} = O(\tau^{3+2i\nu} e^{\frac{1}{2}\nu\pi})$, $I_{31} = O(\tau^3 e^{-\frac{1}{2}\nu\pi})$ with the result that $\mathscr{R}_{31} = O(\tau^{3+2i\nu})$, $\mathscr{T}_{31} = O(\tau^3 e^{-\nu\pi})$ so that the transmitted wave is still negligible though the reflected wave is building up from its value of order $e^{-\nu\pi}$.

8. Explicit calculation of the second-order terms

In this section we evaluate Ψ_{20}, which, it emerges, has a forcing effect on the outer flow, and show that $I_{22} = J_{22} = I_{33} = J_{33} = 0$. Had I_{22}, J_{22} been non-zero a second harmonic would have been forced at this stage on the outer flow. We have extended the obvious definition of reflexion and transmission coefficients to include these terms though it could be argued that the higher harmonics should not be regarded as reflexions of the original imposed wave. The second harmonic is in fact postponed until the I_{42}, J_{42} stage.

As in I we first look at the computation of

$$\Psi_2 = e^{2i\alpha x}\Psi_{22} + \Psi_{20} + e^{-2i\alpha x}\tilde{\Psi}_{22} \tag{8.1}$$

and

$$S_2 = e^{2i\alpha x}S_{22} + S_{20} + e^{-2i\alpha x}\tilde{S}_{22}. \tag{8.2}$$

It follows from (6.2), (6.3) that

$$\frac{\partial \Psi_{20}}{\partial \tau} = i\left(\Psi_{11}\frac{\partial \tilde{\Psi}_{11}}{\partial Y} - \tilde{\Psi}_{11}\frac{\partial \Psi_{11}}{\partial Y}\right), \quad \frac{\partial S_{20}}{\partial \tau} = i\frac{\partial}{\partial Y}(\Psi_{11}\tilde{S}_{11} - \Psi_{11}S_{11}) \tag{8.3}$$

so that, from (6.5),

$$\frac{\partial \Psi_{20}}{\partial \tau} = -|b|^2 \int_0^{\tau}\!\!\int_0^{\tau} \frac{(u+v)\,e^{-iY(u-v)}}{u^{\frac{3}{2}-i\nu}v^{\frac{3}{2}+i\nu}}\,du\,dv. \tag{8.4}$$

A useful expression for $\partial S_{20}/\partial \tau$ is then obtained, on use of (6.10), as

$$\frac{\partial S_{20}}{\partial \tau} = \frac{1}{2J}\frac{\partial^2 \Psi_{20}}{\partial \tau\,\partial Y} + \frac{\nu}{J}\frac{\partial^2}{\partial Y^2}(\Psi_{11}\tilde{\Psi}_{11}). \tag{8.5}$$

The behaviour of Ψ_{20} as $|Y| \to \infty$ is most easily obtained from (8.3) and the form

$$\Psi_{11} = b(-\tfrac{3}{2}+i\nu)!\left\{e^{\pm\frac{1}{4}i\pi\pm\frac{1}{2}\nu\pi}|Y|^{\frac{1}{2}-i\nu} + \frac{\cosh \nu\pi}{\pi}\int_0^{\infty}\frac{u^{\frac{1}{2}-i\nu}\,e^{-(iY+u)\tau}}{u+iY}\,du\right\} \tag{8.6}$$

according as $Y \gtrless 0$. Using this we find that

$$\Psi_{20} \approx -2\nu\tau + 2\nu^2/Y + O\{\tau e^{-iY\tau}(Y\tau)^{-\frac{3}{2}+i\nu}\}, \tag{8.7}$$

if $Y > 0$ and

$$\Psi_{20} \approx 2\nu\tau e^{-2\nu\pi} - \nu/(\pi|Y|) + O\{\tau e^{-iY\tau}|Y\tau|^{-\frac{3}{2}+i\nu}\}, \tag{8.8}$$

if $Y < 0$. Thus Ψ_{20} does not decay to zero as $|Y| \to \infty$ and the stream function in the outer region where $y = O(1)$ has forced on it an x- and y-independent term that is $O(\epsilon t)$. The presence of this term in the outer solution in no way invalidates the remarks made early in § 7 about the relative orders of magnitude of the neglected terms in (7.1).

From (8.4) it also follows that, when $\nu \gg 1$,

$$\left.\frac{\partial \Psi_{20}}{\partial Y}\right|_{Y=0} = -\frac{2|b|^2\tau^2}{\nu^3}, \quad \left.\frac{\partial^2 \Psi_{20}}{\partial Y^2}\right|_{Y=0} = -\frac{4|b|^2\tau^3}{\nu^4}. \tag{8.9}$$

Since for a comparison with the situation when the imposed wave is below the critical layer we must change the sign of Y but not of Ψ, the sign of the first of these is not in accord with the prediction of Ramanathan & Cess (1975) on the mean retrograde winds within the atmosphere of Venus. Also that of the second does not support the findings of Lindzen & Rosenthal (1976) on the sharpening of the mean shear. However it is noted that, although broadly negative, $\partial\Psi_{20}/\partial Y$ does oscillate in Y.

To proceed further we must calculate Ψ_{22}, S_{22}. These satisfy (6.12), (6.13) with $r = n = 2$ and

$$H_{22}(Y,\tau) = i\left(\Psi_{11}\frac{\partial S_{11}}{\partial Y} - S_{11}\frac{\partial \Psi_{11}}{\partial Y}\right) = \frac{i}{\frac{1}{2}-i\nu}\left\{\Psi_{11}\frac{\partial^2\Psi_{11}}{\partial Y^2} - \left(\frac{\partial\Psi_{11}}{\partial Y}\right)^2\right\},$$

$$G_{22}(Y,\tau) = i\left(\Psi_{11}\frac{\partial^3\Psi_{11}}{\partial Y^3} - \frac{\partial^2\Psi_{11}}{\partial Y^2}\frac{\partial\Psi_{11}}{\partial Y}\right) = (\tfrac{1}{2}-i\nu)\frac{\partial H_{22}}{\partial Y}. \tag{8.10}$$

On taking the Laplace transform it is easily verified that

$$\overline{\Psi}_{22}(Y,s) = (\tfrac{1}{2}-i\nu)(s+2iY)^{\frac{1}{2}-i\nu}\int_{-\infty}^{Y}\overline{H}_{22}(Y_1,s)(s+2iY_1)^{-\frac{3}{2}+i\nu}\,dY_1, \tag{8.11}$$

and

$$\overline{S}_{22}(Y,s) = (\tfrac{1}{2}-i\nu)^{-1}\frac{\partial\overline{\Psi}_{22}}{\partial Y}, \tag{8.12}$$

where the neglect of the complementary functions has anticipated that I_{22}, J_{22} will be zero. From the absence of a term in $(s+2iY)^{\frac{1}{2}+i\nu}$ in (8.11) we see immediately that $J_{22} = 0$. Also from (8.11) we obtain

$$I_{22} = \frac{1}{(-\frac{1}{2}-i\nu)!}\int_0^{\tau}(\tau-\tau_2)^{\frac{1}{2}-i\nu}\,d\tau_2\int_{-\infty}^{\infty}H_{22}(Y,\tau_2)\,e^{-2iY(\tau-\tau_2)}\,dY, \tag{8.13}$$

which on substitution for H_{22} becomes

$$I_{22} = \frac{ib^2}{(\frac{1}{2}-i\nu)!}\int_0^{\tau}(\tau-\tau_2)^{\frac{1}{2}-i\nu}\,d\tau_2\,{}^*\!\int_0^{\tau_2}\int_0^{\tau_2}\frac{v(u-v)}{(uv)^{\frac{1}{2}-i\nu}}\,du\,dv\int_{-\infty}^{\infty}e^{-iY(u+v+2\tau-2\tau_2)}\,dY. \tag{8.14}$$

This integral vanishes for exactly the same reasons as did the corresponding integral in I. The inner integral is $2\pi\delta(u+v+2\tau-2\tau_2)$, where δ is a Dirac delta function which vanishes except when $\tau = \tau_2, u = v = 0$, i.e. on the boundary of the hypervolume of integration. However the presence of the factor $(\tau-\tau_2)^{\frac{1}{2}-i\nu}$ in the integrand implies that $I_{22} = 0$.

Later we shall need Ψ_{22} which is obtained by inverting $\overline{\Psi}_{22}$ in (8.11) as

$$\Psi_{22} = -\frac{ib^2}{(\frac{1}{2}-i\nu)!\,(-\frac{3}{2}+i\nu)!}\int_0^{\tau}d\tau_1\int_0^{\tau-\tau_1}d\tau_2\int_0^{\tau_2}\int_0^{\tau_2}\frac{v(u-v)(\tau-\tau_1-\tau_2)^{\frac{1}{2}-i\nu}}{(uv\tau_1)^{\frac{3}{2}-i\nu}}\,du\,dv$$

$$\times\int_0^{\infty}\exp\left[-i\alpha Y(2\tau-2\tau_2+u+v)-i\alpha Y_1(2\tau_1+2\tau_2-2\tau-u-v)\right]dY_1. \tag{8.15}$$

The third-order terms take the form

$$\Psi_{33}\,e^{3i\alpha x}+\Psi_{31}\,e^{i\alpha x}+\text{c.c.} \tag{8.16}$$

with a similar expression involving S_{33}, S_{31}. The terms Ψ_{33}, S_{33} have no effect on the outer flow since it can be shown by writing them down explicitly that $I_{33} = 0 = J_{33}$. The reason is the same as for the vanishing of I_{22}, J_{22}. To obtain the forcing term on the right-hand side of the equation there are no mixtures of for example Ψ_{11} and its complex conjugate $\tilde{\Psi}_{11}$. Thus the delta function from the first stage of the integration is only non-zero on the boundary of the hypervolume of integration where other factors of the integrand vanish. However this does not hold for Ψ_{31}, S_{31}, which lead to non-zero I_{31}, J_{31}; we proceed to calculate these in the following section. These quantities furnish the first correction to the reflexion and transmission coefficients.

9. The first correction of the reflexion and transmission coefficients

In this section we calculate I_{31}, J_{31}, which upon use of (7.11), (7.12) give $\mathscr{R}_{31}, \mathscr{T}_{31}$. We find that $I_{31} = O(\nu^{\frac{3}{2}} e^{-\frac{1}{2}\nu\pi})$, $J_{31} = O(\nu^{-1} e^{\frac{1}{2}\nu\pi})$ when ν is large so that $\mathscr{R}_{31} = O(\nu^{-1})$, $\mathscr{T}_{31} = O(\nu^{\frac{3}{2}} e^{-\nu\pi})$. The τ dependence of $\mathscr{R}_{31}$ is $\tau^{3+2i\nu}$, and of $\mathscr{T}_{31}$ is τ^3. Thus the transmission coefficient has the same dominant dependence as its steady value (5.6) though the reflexion coefficient has a correction that is much larger in magnitude.

The equations satisfied by Ψ_{31}, S_{31} are

$$\left(\frac{\partial}{\partial\tau}+iY\right)\frac{\partial^2\Psi_{31}}{\partial Y^2}+iJS_{31} = \frac{\partial M_{31}}{\partial Y}, \quad \left(\frac{\partial}{\partial\tau}+iY\right)S_{31}-i\Psi_{31} = H_{31}, \tag{9.1}$$

where

$$M_{31}(Y,\tau) = i\left(\Psi_{11}\frac{\partial^2\Psi_{20}}{\partial Y^2}-\frac{\partial\Psi_{11}}{\partial Y}\frac{\partial\Psi_{20}}{\partial Y}\right)+i\left(2\Psi_{22}\frac{\partial^2\tilde{\Psi}_{11}}{\partial Y^2}-\frac{\partial\tilde{\Psi}_{11}}{\partial Y}\frac{\partial\Psi_{22}}{\partial Y}-\tilde{\Psi}_{11}\frac{\partial^2\Psi_{22}}{\partial Y^2}\right), \tag{9.2}$$

$$H_{31}(Y,\tau) = i\left(\Psi_{11}\frac{\partial S_{20}}{\partial Y}-S_{11}\frac{\partial\Psi_{20}}{\partial Y}\right)+i\left(2\Psi_{22}\frac{\partial\tilde{S}_{11}}{\partial Y}-2S_{22}\frac{\partial\tilde{\Psi}_{11}}{\partial Y}+\tilde{S}_{11}\frac{\partial\Psi_{22}}{\partial Y}-\frac{\partial S_{22}}{\partial Y}\tilde{\Psi}_{11}\right). \tag{9.3}$$

The solution of (9.1) for $\bar{\Psi}_{31}$ is

$$\begin{aligned}\bar{\Psi}_{31} = {} & \frac{i}{2\nu}(s+iY)^{\frac{1}{2}-i\nu}\int_{-\infty}^{Y}\{(\tfrac{1}{2}-i\nu)\bar{M}_{31}(Y_1,s)-J\bar{H}_{31}(Y_1,s)\}(s+iY_1)^{-\frac{3}{2}+i\nu}\,dY_1 \\ & -\frac{i}{2\nu}(s+iY)^{\frac{1}{2}+i\nu}\int_{-\infty}^{Y}\{(\tfrac{1}{2}+i\nu)\bar{M}_{31}(Y_1,s)-J\bar{H}_{31}(Y_1,s)\}(s+iY_1)^{-\frac{3}{2}-i\nu}\,dY_1 \\ & +\bar{C}_{31}(s+iY)^{\frac{1}{2}-i\nu}+\bar{D}_{31}(s+iY)^{\frac{1}{2}+i\nu},\end{aligned} \tag{9.4}$$

so that

$$I_{31} = \frac{i}{2\nu(\frac{1}{2}-i\nu)!}\int_0^\tau(\tau-\tau_3)^{\frac{1}{2}-i\nu}\,d\tau_3\int_{-\infty}^{\infty}\{(\tfrac{1}{2}-i\nu)M_{31}(Y,\tau_3)-JH_{31}(Y,\tau_3)\}e^{-iY(\tau-\tau_3)}\,dY, \tag{9.5}$$

$$J_{31} = \frac{i}{2\nu(\frac{1}{2}+i\nu)!}\int_0^\tau(\tau-\tau_3)^{\frac{1}{2}+i\nu}\,d\tau_3\int_{-\infty}^{\infty}\{(\tfrac{1}{2}+i\nu)M_{31}(Y,\tau_3)-JH_{31}(Y,\tau_3)\}e^{-iY(\tau-\tau_3)}\,dY. \tag{9.6}$$

It is convenient to split I_{31}, J_{31} into two parts and write $I_{31} = I_{310}+I_{312}, J_{31} = J_{310}+J_{312}$ and similarly with M_{31}, H_{31}. In (9.2), (9.3) the first bracket corresponds to the subscript 0 and the second to the subscript 2. Thus for example

$$M_{310}(Y,\tau) = i\left(\Psi_{11}\frac{\partial^2\Psi_{20}}{\partial Y^2}-\frac{\partial\Psi_{11}}{\partial Y}\frac{\partial\Psi_{20}}{\partial Y}\right). \tag{9.7}$$

To compute (9.5), (9.6) we first calculate these functions and obtain

$$\begin{aligned}(\tfrac{1}{2}-i\nu)M_{310}-JH_{310} = {} & -2i\nu\Psi_{11}\int_0^\tau\frac{\partial^2}{\partial Y^2}\left(\tilde{\Psi}_{11}\frac{\partial\Psi_{11}}{\partial Y}\right)d\tau_1 \\ & -2i\nu\frac{\partial\Psi_{11}}{\partial Y}\int_0^\tau\left(\Psi_{11}\frac{\partial^2\tilde{\Psi}_{11}}{\partial Y^2}-\tilde{\Psi}_{11}\frac{\partial^2\Psi_{11}}{\partial Y^2}\right)d\tau_1,\end{aligned} \tag{9.8}$$

$$(\tfrac{1}{2}+i\nu)M_{310}-JH_{310} = -2i\nu\Psi_{11}\int_0^\tau\frac{\partial^2}{\partial Y^2}\left(\Psi_{11}\frac{\partial\tilde{\Psi}_{11}}{\partial Y}\right)d\tau_1, \tag{9.9}$$

$$(\tfrac{1}{2}-i\nu)\,M_{312}-JH_{312} = -2\nu\tilde{\Psi}_{11}\frac{\partial^2\Psi_{22}}{\partial Y^2}-4\nu\frac{\partial\tilde{\Psi}_{11}}{\partial Y}\frac{\partial\Psi_{22}}{\partial Y}, \tag{9.10}$$

$$(\tfrac{1}{2}+i\nu)\,M_{312}-JH_{312} = -2\nu\frac{\partial\tilde{\Psi}_{11}}{\partial Y}\frac{\partial\Psi_{22}}{\partial Y}-4\nu\frac{\partial^2\tilde{\Psi}_{11}}{\partial Y^2}\Psi_{22}, \tag{9.11}$$

where use has been made of (8.3), (8.5) and (8.12).

From (9.5), (9.8) we are able to write I_{310} as the multiple integral

$$I_{310} = \frac{ib|b|^2}{(\frac{1}{2}-i\nu)!}\int_0^\tau (\tau-\tau_3)^{\frac{1}{2}-i\nu}\,d\tau_3\int_0^{\tau_3}d\tau_1\int_0^{\tau_1}\int_0^{\tau_1}du\,dv\int_0^{\tau_3}dw$$
$$\times\frac{(v-u)\{u(v-u)+w(u+v)\}}{(uw)^{\frac{3}{2}-i\nu}\,v^{\frac{3}{2}+i\nu}}\int_{-\infty}^{\infty}\exp\left[-iY(\tau-\tau_3+u+w-v)\right]dY \tag{9.12}$$

and from (9.6), (9.9) we have

$$J_{310} = -\frac{ib|b|^2}{(\frac{1}{2}+i\nu)!}\int_0^\tau (\tau-\tau_3)^{\frac{1}{2}+i\nu}\,d\tau_3\int_0^{\tau_3}d\tau_1\int_0^{\tau_1}\int_0^{\tau_1}du\,dv\int_0^{\tau_3}dw$$
$$\times\frac{(v-u)^2}{(uw)^{\frac{3}{2}-i\nu}\,v^{\frac{1}{2}+i\nu}}\int_{-\infty}^{\infty}\exp\left[-iY(\tau-\tau_3+u+w-v)\right]dY. \tag{9.13}$$

Evaluation of both of these integrals is facilitated by use of the delta function. In both cases the innermost integral yields $2\pi\delta(\tau-\tau_3+u+w-v)$ which is non-zero only at $v = u+w+\tau-\tau_3$ and this point must therefore be interior to the polyhedron over which the remaining fivefold integral is taken. Then J_{310} for example reduces to

$$J_{310} = -\frac{2\pi ib|b|^2}{(\frac{1}{2}+i\nu)!}\int_{\frac{1}{2}\tau}^{\tau}(\tau-\tau_3)^{\frac{1}{2}+i\nu}\,d\tau_3\int_{\tau-\tau_3}^{\tau_3}d\tau_1\iint^{u+w\leqslant\tau_1+\tau_3-\tau}\frac{(v-u)^2\,du\,dw}{(uw)^{\frac{3}{2}-i\nu}\,v^{\frac{1}{2}+i\nu}} \tag{9.14}$$

with $v = u+w+\tau-\tau_3$. The corresponding form for I_{310} may be written down similarly. The integrals may now be performed in the order indicated and the results are

$$I_{310} = \tfrac{8}{3}\pi ib|b|^2\tau^3\frac{(-\frac{1}{2}+i\nu)!\,(-\frac{3}{2}+i\nu)!}{(\frac{1}{2}-i\nu)!\,(2i\nu)!}$$
$$\times\left\{\int_0^1\frac{(1-p)^{2i\nu}p^{\frac{1}{2}-i\nu}}{(1+p)^2}\,dp+\frac{-\frac{1}{2}+i\nu}{\frac{1}{2}+i\nu}\int_0^1\frac{(1-p)^{1+2i\nu}p^{\frac{1}{2}-i\nu}}{(1+p)^3}\,dp\right\}, \tag{9.15}$$

$$J_{310} = \frac{4\pi ib|b|^2\tau^{3+2i\nu}}{3+2i\nu}\,\frac{(-\frac{3}{2}+i\nu)!}{(\frac{1}{2}-i\nu)\,(2i\nu)!}$$
$$\times\left\{\int_0^1\frac{(1-p)^{2i\nu}p^{\frac{1}{2}+i\nu}}{(1+p)^{3+2i\nu}}\,dp-\frac{\frac{3}{2}+i\nu}{\frac{1}{2}+i\nu}\int_0^1\frac{(1-p)^{2i\nu}p^{\frac{3}{2}+i\nu}}{(1+p)^{4+2i\nu}}\,dp\right\}. \tag{9.16}$$

The integrals for I_{312}, J_{312} are eightfold and are, on use of (8.15) for Ψ_{22},

$$I_{312} = \frac{4b|b|^2}{(\frac{1}{2}-i\nu)!\,(\frac{1}{2}-i\nu)!\,(-\frac{3}{2}+i\nu)!}\int_0^\tau d\tau_3\int_0^{\tau_3}d\tau_1\int_0^{\tau_3-\tau_1}d\tau_2\int_0^{\tau_2}\int_0^{\tau_2}du\,dv\int_0^{\tau_3}dw$$
$$\times\frac{v(u-v)\,(\tau_3-\tau_1-\tau_2)^{\frac{1}{2}-i\nu}\,(\tau-\tau_3)^{\frac{1}{2}-i\nu}}{(uv\tau_1)^{\frac{3}{2}-i\nu}\,w^{\frac{3}{2}+i\nu}}\{\tau_3-\tau_2+\tfrac{1}{2}(u+v)\}\{\tau_3-\tau_2+\tfrac{1}{2}(u+v)-w\}$$
$$\times\int_0^\infty\exp\left[-iY_1(2\tau_1+2\tau_2-2\tau_3-u-v)\right]dY_1$$
$$\times\int_{-\infty}^{\infty}\exp\left[-iY(\tau+\tau_3-2\tau_2+u+v-w)\right]dY, \tag{9.17}$$

$$J_{312} = \frac{2b|b|^2}{(\frac{1}{2}-i\nu)!\,(\frac{1}{2}+i\nu)!\,(-\frac{3}{2}+i\nu)!}\int_0^\tau d\tau_3 \int_0^{\tau_3} d\tau_1 \int_0^{\tau_3-\tau_1} d\tau_2 \int_0^{\tau_2}\int_0^{\tau_2} du\,dv \int_0^{\tau_3} dw$$
$$\times \frac{v(u-v)\,(\tau_3-\tau_1-\tau_2)^{\frac{1}{2}-i\nu}\,(\tau-\tau_3)^{\frac{1}{2}+i\nu}}{(uv\tau_1)^{\frac{3}{2}-i\nu}\,w^{\frac{1}{2}+i\nu}}\,(2w-2\tau_3+2\tau_2-u-v)$$
$$\times \int_0^\infty \exp\left[-iY_1(2\tau_1+2\tau_2-2\tau_3-u-v)\right]dY_1$$
$$\times \int_{-\infty}^\infty \exp\left[-iY(\tau+\tau_3-2\tau_2+u+v-w)\right]dY. \quad (9.18)$$

The two inner integrals may be evaluated immediately on use of the delta function to give

$$I_{312} = \frac{-4\pi i b|b|^2}{(\frac{1}{2}-i\nu)!\,(\frac{1}{2}-i\nu)!\,(-\frac{3}{2}+i\nu)!}\int_{\frac{1}{2}\tau}^\tau d\tau_3 \int_0^{\tau_3-\frac{1}{2}\tau} d\tau_1 \int_{\frac{1}{2}\tau}^{\tau_3-\tau_1} d\tau_2 \iint^{u+v\leqslant 2\tau_2-\tau} du\,dv$$
$$\times \frac{v(u-v)\,(\tau_3-\tau_1-\tau_2)^{\frac{1}{2}-i\nu}\,(\tau-\tau_3)^{\frac{1}{2}-i\nu}\{\tau_3-\tau_2+\frac{1}{2}(u+v)\}\{\tau_2-\tau-\frac{1}{2}(u+v)\}}{(uv\tau_1)^{\frac{3}{2}-i\nu}\,w^{\frac{3}{2}+i\nu}\{\tau_1+\tau_2-\tau_3-\frac{1}{2}(u+v)\}}, \quad (9.19)$$

and

$$J_{312} = \frac{-2\pi i b|b|^2}{(\frac{1}{2}-i\nu)!\,(\frac{1}{2}+i\nu)!\,(-\frac{3}{2}+i\nu)!}\int_{\frac{1}{2}\tau}^\tau d\tau_3 \int_0^{\tau_3-\frac{1}{2}\tau} d\tau_1 \int_{\frac{1}{2}\tau}^{\tau_3-\tau_1} d\tau_2 \iint^{u+v\leqslant 2\tau_2-\tau} du\,dv$$
$$\times \frac{v(u-v)\,(\tau_3-\tau_1-\tau_2)^{\frac{1}{2}-i\nu}\,(\tau-\tau_3)^{\frac{1}{2}+i\nu}\{\tau-\tau_2+\frac{1}{2}(u+v)\}}{(uv\tau_1)^{\frac{3}{2}-i\nu}\,w^{\frac{1}{2}+i\nu}\{\tau_1+\tau_2-\tau_3-\frac{1}{2}(u+v)\}}, \quad (9.20)$$

where in both cases $w = \tau-2\tau_2+\tau_3+u+v$.

The inner pair of integrals is reduced to a single integral on use of the result that

$$\iint^{u+v\leqslant 2\tau_2-\tau}\left\{\frac{1}{(uv)^{\frac{1}{2}-i\nu}}-\frac{v^{\frac{1}{2}+i\nu}}{u^{\frac{3}{2}-i\nu}}\right\}f(u+v)\,du\,dv$$
$$= \frac{(-\frac{1}{2}+i\nu)!\,(-\frac{1}{2}+i\nu)!}{(\frac{1}{2}-i\nu)\,(2i\nu)!}\int_0^{2\tau_2-\tau} x^{2i\nu}f(x)\,dx \quad (9.21)$$

and after that it is helpful to make the successive substitutions

$$x = 2t_2-2y, \quad \tau_2 = \tfrac{1}{2}\tau+t_2, \quad \tau_3 = \tfrac{1}{2}\tau+t_3, \quad \tau_1 = t_3-t_1$$

and then it is possible to perform the integrals with suitable choice of the order. The final result is

$$I_{312} = \tfrac{1}{3}\pi i b|b|^2 2^{2i\nu}\tau^3\,\frac{(-\frac{1}{2}+i\nu)!\,(-\frac{3}{2}+i\nu)!}{(\frac{3}{2}+i\nu)\,(\frac{1}{2}-i\nu)!\,(2i\nu)!}\int_0^1 \frac{p^{2i\nu+1}(1-p)^{\frac{1}{2}-i\nu}}{(1+p)^{\frac{3}{2}+i\nu}}\,dp, \quad (9.22)$$

and

$$J_{312} = \frac{\pi i b|b|^2\tau^{3+2i\nu}(-\frac{1}{2}+i\nu)!}{(3+2i\nu)^2\,(\frac{1}{4}+\nu^2)\,(2i\nu)!}\int_0^1 \frac{p^{2i\nu}(1-p)^{\frac{1}{2}+i\nu}}{(1+p)^{\frac{1}{2}+i\nu}}\,dp. \quad (9.23)$$

Apart from the neglect of the term multiplying B_1 in (5.2) these results are exact and we now examine the form of I_{31}, J_{31} when ν is large. We first need the asymptotic behaviour of the six integrals in (9.15), (9.16), (9.22), (9.23) when $\nu \gg 1$. This is achieved by noting that each integral is a hypergeometric function $F(a,b,c,-1)$ for appropriate a, b, c and then using the differential equation satisfied by $F(a, b, c, x)$. The results are

$$\frac{(\frac{3}{2}+i\nu)!}{(\frac{1}{2}-i\nu)!\,(2i\nu)!}\int_0^1 \frac{(1-p)^{2i\nu}\,p^{\frac{1}{2}-i\nu}}{(1+p)^2}\,dp \approx \frac{i\nu}{2}, \tag{9.24}$$

$$\frac{(\frac{5}{2}+i\nu)!}{(\frac{1}{2}-i\nu)!\,(2i\nu+1)!}\int_0^1 \frac{(1-p)^{1+2i\nu}\,p^{\frac{1}{2}-i\nu}}{(1+p)^3}\,dp \approx \tfrac{1}{8}e^{\frac{3}{4}i\pi}\pi^{\frac{1}{2}}\nu^{\frac{3}{2}}, \tag{9.25}$$

$$\frac{(\frac{3}{2}+3i\nu)!}{(\frac{1}{2}+i\nu)!\,(2i\nu)!}\int_0^1 \frac{(1-p)^{2i\nu}\,p^{\frac{1}{2}+i\nu}}{(1+p)^{3+2i\nu}}\,dp \approx \frac{3^{3i\nu+2}2^{3i\nu-\frac{3}{2}}}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{3}{2}}}, \tag{9.26}$$

$$\frac{(\frac{5}{2}+3i\nu)!}{(\frac{3}{2}+i\nu)!\,(2i\nu)!}\int_0^1 \frac{(1-p)^{2i\nu}\,p^{\frac{3}{2}+i\nu}}{(1+p)^{4+2i\nu}}\,dp \approx \frac{3^{3i\nu+3}2^{3i\nu-\frac{1}{2}}}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{7}{2}}}, \tag{9.27}$$

$$\frac{(\frac{5}{2}+i\nu)!}{(\frac{1}{2}-i\nu)!\,(2i\nu+1)!}\int_0^1 \frac{p^{2i\nu+1}(1-p)^{\frac{1}{2}-i\nu}}{(1+p)^{\frac{3}{2}+i\nu}}\,dp \approx \frac{i\nu}{2^{2i\nu+1}}, \tag{9.28}$$

$$\frac{(\frac{3}{2}+3i\nu)!}{(\frac{1}{2}+i\nu)!\,(2i\nu)!}\int_0^1 \frac{p^{2i\nu}(1-p)^{\frac{1}{2}+i\nu}}{(1+p)^{\frac{1}{2}+i\nu}}\,dp \approx \frac{3^{3i\nu+2}2^{3i\nu+\frac{3}{2}}}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{5}{2}}}. \tag{9.29}$$

A final collection of the relevant terms then gives that for large ν

$$I_{31} \approx \frac{2^{-i\nu}}{12}\,\pi^{\frac{1}{2}}\epsilon^{-\frac{2}{3}i\nu}\nu^{\frac{3}{2}}\,e^{-\frac{1}{2}\nu\pi}\tau^3, \tag{9.30}$$

$$J_{31} \approx \frac{2^{2i\nu-\frac{7}{2}}}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{1}{2}}}\,\epsilon^{-\frac{2}{3}i\nu}\nu^{-1-2i\nu}\,e^{-\frac{1}{4}i\pi+2i\nu+\frac{1}{2}\nu\pi}\tau^{3+2i\nu}, \tag{9.31}$$

where use has been made of (6.6), (6.7).

If we now return to (7.11), (7.12) we are in a position to calculate the corrections $\mathscr{R}_{31}, \mathscr{T}_{31}$ to the reflexion and transmission coefficients. On use of (4.11) to substitute for α_{ij}, β_{ij} we obtain that, when $\nu \gg 1$,

$$\mathscr{R}_{31} \approx -\frac{2^{i\nu-\frac{7}{2}}}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{1}{2}}}\,\epsilon^{-\frac{4}{3}i\nu}\nu^{-1-2i\nu}\,e^{2i\nu}\tau^{3+2i\nu}, \tag{9.32}$$

$$\mathscr{T}_{31} \approx -\frac{\pi^{\frac{1}{2}}}{12}\,\nu^{\frac{3}{2}}\,e^{-\frac{1}{4}i\pi-\nu\pi}\tau^3, \tag{9.33}$$

and then from (7.8), (7.9)

$$C_{31} \approx -\frac{\pi^{\frac{1}{2}}}{12}\,2^{-i\nu}\epsilon^{-\frac{2}{3}i\nu}\nu^{\frac{3}{2}}\,e^{-\frac{1}{2}\nu\pi}\tau^3, \tag{9.34}$$

$$D_{31} \approx -\frac{i}{12}\,\pi^{\frac{1}{2}}2^{\frac{1}{2}-i\nu}\epsilon^{\frac{2}{3}i\nu}\nu^{\frac{3}{2}}\,e^{-\frac{5}{2}\nu\pi}\,\tau^3. \tag{9.35}$$

Evaluation and discussion of these results is undertaken in the following section.

10. Results and discussion

We are now in possession of the first correction to the reflexion and transmission coefficients. The results are that, to $O(\tau^3)$, from (5.6), (9.32), (9.33) the reflected and transmitted waves are of the form, when $\nu \gg 1$,

$$\mathscr{R}\, e^{i(\alpha x+my)} + \text{c.c.}, \quad \mathscr{T} e^{i(\alpha x-m|y|)} + \text{c.c.}, \tag{10.1}$$

where

$$\mathscr{R} = -i2^{\frac{1}{2}-2i\nu}\, e^{-\nu\pi} - \frac{2^{i\nu-\frac{7}{2}}}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{1}{2}}}\, \epsilon^{-\frac{4}{3}i\nu}\, e^{2i\nu} \nu^{-1-2i\nu} \tau^{3+2i\nu}, \tag{10.2}$$

$$\mathscr{T} = -i\, e^{-\nu\pi} - \frac{\pi^{\frac{1}{2}}}{12} e^{-\frac{1}{4}i\pi-\nu\pi} \nu^{\frac{3}{2}} \tau^3. \tag{10.3}$$

Thus at this stage the transmitted wave is still $O(e^{-\nu\pi})$ for ν large and so there is again no transmission through the critical layer. However the coefficient of the reflected wave is, as a function of ν, $O(\nu^{-1} e^{\nu\pi})$ larger at the $O(\tau^3)$ stage than at the $O(1)$ stage. As time goes on the critical layer which when $t = O(1)$ acted merely as an absorber of energy increasingly takes on the role of a reflector and returns some of the energy to the main region of the flow. The increase in size of the transmission coefficient is only $O(\nu^{\frac{3}{2}})$.

It is interesting to speculate on the outcome of a continuation of the calculation, formidable though it would be. So far our achievement is limited, being confined to the computation of the leading correction to the prescribed harmonic. Higher harmonics will be generated in the outer region of the flow but there is no second harmonic $O(\tau^{\frac{3}{2}})$ because I_{22}, J_{22} happened to be zero. Similarly there is no third harmonic $O(\tau^3)$. However, by analogy with I, there is no doubt that a second harmonic will be generated at the $O(\tau^{\frac{9}{2}})$ stage. There will also be corrections of order $\tau^6, \tau^{\frac{15}{2}}$ etc. to the first harmonic. It seems, though, that it is of dubious value to attempt to estimate the size of the coefficients as functions of ν when $\nu \gg 1$. It is the subtle interplay of the signs of $i\nu$ that led to J_{31} being $O(e^{\nu\pi})$ larger than I_{31}. An illustration of this is given by the beta functions

$$\int_0^1 y^{-\frac{1}{2}-i\nu}(1-y)^{-\frac{1}{2}+i\nu}\, dy \quad \text{and} \quad \int_0^1 y^{-\frac{1}{2}-i\nu}(1-y)^{-\frac{1}{2}-i\nu}\, dy, \tag{10.4}$$

the values of which are $\pi \operatorname{sech} \nu\pi$ and $\pi^{\frac{1}{2}} 2^{\frac{3}{2}+i\nu}(-\frac{1}{2}-i\nu)!/(-i\nu)!$ respectively, so that their ratio is $O(\nu^{-\frac{1}{2}} e^{-\nu\pi})$ when ν is large. In fact I_{rn}, J_{rn} consist of multiple products of terms of this form. One might suspect however that later terms in the expansion will give a contribution to the transmission coefficient that is not exponentially small in ν since $\overline{\Psi}_{31}$ now has a term which takes the form $J_{31}\, (s+iY)^{\frac{1}{2}+i\nu}$ when Y is large and positive, where $J_{31} = O(e^{\frac{1}{2}\nu\pi})$ and this could generate an $O(1)$ contribution to the transmission coefficient at the $O(\tau^9)$ stage. However a separate study indicates that this is deferred until the $O(\tau^{12})$ stage.

We conclude by noting that at points of overlap our work is in agreement with that of previous authors. At times when the linear theory is valid the critical layer is, for Richardson numbers sensibly greater than $\frac{1}{4}$, a wave absorber. On longer time scales when the nonlinear terms become important the critical layer starts to return energy to the outer flow, the mechanism being the reflected wave. The calculation has not been carried sufficiently far here to see if there will also be a non-negligible transmitted wave, nor have any higher harmonics been explicitly calculated though these will certainly occur. Both these tasks have, however, been undertaken in a subsequent study.

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